$$\frac{c}{L}e_{yx}^{(1)} = O(\varepsilon_x^{(1)}), \quad \frac{c}{L}k_y^{(1)} = O(K_x^{(1)}), \quad \frac{c}{L}\sigma_{xy}^{(1)} = O(\sigma_{xx}^{(1)}) \qquad (33)$$

Equations (33) will be found to imply the requirement of certain stipulations concerning admissible relative orders of magnitudes of the coefficient functions E, G, Γ and ν. At the same time, more stringent types of stipulations (such as for example that $E_x = O(G)$) would allow us the use of a first-step equation system without some of the terms retained in (29) (without affecting the validity of the results obtained through use of equations (29) as they stand).

Finally, we note the following property of the equations of the iterative procedure above. For the case that all elasticity coefficient functions E, G, ν and Γ are independent of the coordinate x, that is for the case corresponding to equations (6) to (13), the terms omitted in going from equations (27) to equations (29) happen to be those x-derivative terms which vanish in the exact solution. As a consequence, for this special case the results of the first step of the iterative procedure, in the maximally complete form (29), would not be modified by the subsequent steps of the procedure.

ON THE STABILITY OF THREE-DIMENSIONAL ELASTIC BODIES

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The static stability of a three-dimensional elastic body with small subcritical strains is considered. Because of the assumption of smallness of the subcritical strains the results obtained below are applicable to the investigation of the stability of elastic bodies fabricated from a metal and from stiff bonded plastics. These results are also necessary for the latter since bonded plastics have low shear stiffness, hence application of the applied theories sometimes results in large errors in determining the critical forces.

Special attention is paid to obtaining general solutions of the static stability equations of a three-dimensional body compressed along the Ox_3 axis by stress resultants of intensity q, and along the Ox_1 and Ox_2 axes by stress resultants of intensity p. In the particular case of $p = 0$, solutions of a similar form [1] permitted the investigation of the stability of cylindrical shells [2] and bars [3]. The first members of the asymptotic expansions of the magnitudes of the critical force, which agree with the value of the critical force obtained with the aid of the Kirchhoff-Love hypothesis, were calculated in [2] and [3].

General solutions in invariant form are constructed below, which permit the investigation of the stability of hollow cylindrical shells, and of shells with a filler, of bars, of plates both single and multilayered subjected to the loadings mentioned above. As an illustration, the stability of rectangular and circular plates under multilateral compression is considered, where the boundary conditions are satisfied approximately in the integral sense.

Let us consider the static stability of a three-dimensional body with small subcritical strains compressed by stress resultants of intensity q along the x_3 axis and by stress resultants of intensity p along the x_1 and x_2 axes. The fundamental variational equations can be represented, following [4–7], as

$$[\sigma_{im} - q\delta_{i3}u_{m,3} - p(\delta_{i1}u_{m,1} + \delta_{i2}u_{m,2})]_{,i} = 0 \tag{1}$$

Let us consider the body to be transversely isotropic with x_3 the axis of isotropy, then the connection between the stresses and strains can be represented as follows:

$$\sigma_{ij} = \delta_{ij}a_{ik}u_{k,k} + (1 - \delta_{ij})G_{ij}(u_{i,j} + u_{j,i}) \tag{2}$$

$$a_{11} = a_{22}, \qquad a_{13} = a_{23}, \qquad G_{12} = {}^1/_2\,(a_{11} - a_{12}), \qquad G_{13} = G_{23} = G$$

It must be noted that the symmetry in the elastic properties here corresponds to symmetry of the fundamental state of stress (subcritical state). This circumstance affords the possibility of including nonlinearly elastic media in the considerations since the linearized law connecting the stresses and strains (more accurately, their increments) in the considered state of stress can be represented by (2). Accepting the concept of the tangent modulus, we can also include stability under small elastoplastic strains in the considerations. Therefore, the results obtained below refer equally to nonlinearly elastic bodies, and to elastoplastic bodies, except that it is necessary to determine the values of the quantities a_{ij} and G by starting from the specific form of the relationship between the stresses and strains.

Substituting (2) into (1), we obtain the equations in displacement

$$L_{mj}u_j = 0 \qquad (m = 1, 2, 3) \tag{3}$$

$$L_{mj} = \delta_{im}a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + (1 - \delta_{jm})\,G_{jm}\frac{\partial^2}{\partial x_m \partial x_j} + \tag{4}$$
$$+ (1 - \delta_{im})\,\delta_{jm}G_{im}\frac{\partial^2}{\partial x_i^2} - (q\delta_{i3}\delta_{n3} + p\delta_{i1}\delta_{n1} + p\delta_{i2}\delta_{n2})\,\delta_{mj}\frac{\partial^2}{\partial x_n \partial x_i}$$

We can represent the solution of the system (3) as one of tree modes, or as their linear combination

$$u_i^{(j)} = \frac{\partial \det \| L_{rs} \|}{\partial (L_{ji})}\,\Phi^{(j)} \qquad (i, j = 1, 2, 3) \tag{5}$$

but not summed over j. The functions $\Phi^{(j)}$ are determined from Eq.

$$\det \| L_{rs} \|\, \Phi^{(j)} = 0 \tag{6}$$

We represent the solution of (6) as

$$\Phi^{(j)} = \Phi_1 + \Phi_2 + \Phi_3 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \zeta_i^2\frac{\partial^2}{\partial x_3^2}\right)\Phi_i = 0 \quad (i = 1, 2, 3) \tag{7}$$

and we obtain the following algebraic equation to determine the ζ_i

$$\zeta^6 - \left[(a_{11} - p)\left(\frac{a_{11} - a_{12}}{2} - p\right)(G - p)\right]^{-1}\left\{\left(\frac{a_{11} - a_{12}}{2} - p\right)[-(G + a_{13})^2 +\right.$$
$$\left. + (G - p)(G - q) + (a_{11} - p)(a_{33} - q)] + (a_{11} - p)(G - p)(G - q)\right\}\zeta^4 +$$
$$+ \left[(a_{11} - p)\left(\frac{a_{11} - a_{12}}{2} - p\right)(G - p)\right]^{-1}(G - p)\left[-(G + a_{13})^2 + \right. \tag{8}$$
$$\left. + (a_{33} - q)\left(\frac{a_{11} - a_{12}}{2} - p\right) + (G - p)(G - q) + (a_{33} - q)(a_{11} - p)\right]\zeta^2 -$$
$$- \left[(a_{11} - p)\left(\frac{a_{11} - a_{12}}{2} - p\right)(G - p)\right]^{-1}(a_{33} - q)(G - q)^2 = 0$$

The roots of (8) are

$$\zeta_1^2 = \frac{G - q}{{}^1/_2(a_{11} - a_{12}) - p}, \qquad \zeta_{2,3}^2 = \frac{1}{2}\,\frac{-(a_{13} + G)^2 + (G - p)(G - q) + (a_{11} - p)(a_{33} - q)}{(a_{11} - p)(G - p)} \pm$$

$$\pm\left\{\left[\frac{1}{2}\frac{-(a_{13}+G)^2+(G-p)(G-q)+(a_{11}-p)(a_{33}-q)}{(a_{11}-p)(G-p)}\right]^2-\frac{(G-q)(a_{33}-q)}{(a_{11}-p)(G-p)}\right\}^{1/2} \quad (9)$$

To determine the displacements we put

$$\Phi^{(1)}=\Phi_1,\quad \Phi^{(2)}=\Phi_1,\quad \Phi^{(3)}=\Phi_2+\Phi_3,\quad u_i=u_i^{(1)}-u_i^{(2)}+u_i^{(3)} \quad (10)$$

and we introduce the new functions Ψ_i which satisfy (7) and are connected with the functions Φ_i as follows:

$$\Psi_1=(G-p)\frac{a_{11}+a_{12}}{2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left\{\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\left[\frac{a_{33}-q}{G-p}-\right.\right.$$
$$\left.\left.-2\frac{(a_{13}+G)^2}{(G-p)(a_{11}+a_{12})}\right]\frac{\partial^2}{\partial x_3^2}\right\}\Phi_1 \qquad (i=2,3) \quad (11)$$
$$\Psi_i=(G+a_{13})\left(\frac{a_{11}-a_{12}}{2}-p\right)\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+(G-q)\left(\frac{a_{11}-a_{12}}{2}-p\right)^{-1}\frac{\partial^2}{\partial x_3^2}\right]\Phi_i$$

From the relationships (4), (5) and (9)–(11) we obtain

$$u_1=\frac{\partial}{\partial x_2}\Psi_1-\frac{\partial^2}{\partial x_1\,\partial x_2}(\Psi_2+\Psi_3),\quad u_2=-\frac{\partial}{\partial x_1}\Psi_1-\frac{\partial^2}{\partial x_2\,\partial x_3}(\psi_2+\psi_3)$$

$$u_3=\frac{a_{11}-p}{G+a_{13}}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{G-q}{a_{11}-p}\frac{\partial^2}{\partial x_3^2}\right)(\Psi_2+\Psi_3) \quad (12)$$

We can represent the displacements for a body with curvilinear cross-sectional contour as

$$u_n=\frac{\partial}{\partial s}\Psi_1-\frac{\partial^2}{\partial n\,\partial x_3}(\Psi_2+\Psi_3),\qquad u_s=-\frac{\partial}{\partial n}\Psi_1-\frac{\partial^2}{\partial_s\,\partial x_3}(\Psi_2+\Psi_3)$$

$$u_3=\frac{a_{11}-p}{G+a_{13}}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{G-q}{a_{11}-p}\frac{\partial^2}{\partial x_3^2}\right)(\Psi_2+\Psi_3) \quad (13)$$

If we put $p=0$ in the general solution (7), (9) and (13), we then obtain the general solution for uniaxial compression elucidated in [1].

The boundary conditions for the stresses can be written as [6]

$$P_m=n_i(\sigma_{im}-u_{m,n}[p(\delta_{i1}\delta_{n1}+\delta_{i2}\delta_{n2})+q\delta_{i3}\delta_{n3}] \quad (14)$$

Here P_m are the surface loading components in the body after deformation, n_i are the direction cosines of the normal to the body surface prior to deformation. Let us turn to the investigation of specific problems.

Let us consider inner instability according to the terminology of [8], where inner instability is understood to be the phenomenon in which the structure of the material itself loses stability independently of the kind of boundary conditions. For example, for a laminated material this property is manifest in that warping of the laminae appear in a microvolume. The condition that the system (3) transforms into a system of hyperbolic-elliptic or of hyperbolic type will be the condition for the origination of inner instability in describing such materials within the scope of the theory of homogeneous orthotropic bodies. Let us limit ourselves to the case $a_{33}>G$ and $a_{11}>G$ which can be justified from reasoning of a physical nature. Then, as a result of analysis we obtain the critical value

$$q_*=G \quad (15)$$

Here q_* is independent of p for $p<G$.

Let us examine the stability of a rectangular plate (*) compressed by stress resultants of intensity p along the x_1 and x_2 axes, where the plate thickness is $2h$, the length b and the breadth a. According to (14), the boundary conditions are

*) In the considered problems we assume $q=0$ in (1) and (14) throughout for plates.

$$\sigma_{33} = 0,\ \sigma_{13} = 0,\ \sigma_{23} = 0 \quad \text{for} \quad x_3 = \pm h \tag{16}$$

Let us select the solution of (7) as

$$\Psi_i = A_i \operatorname{ch} \frac{\gamma}{\zeta_i} x_3 \sin \alpha x_1 \sin \beta x_2, \quad \alpha = m \frac{\pi}{a}, \quad \beta = n \frac{\pi}{b} \quad (i=2,3)$$

$$\Psi_1 = A_1 \operatorname{sh} \frac{\gamma}{\zeta_1} x_3 \cos \alpha x_1 \cos \beta x_2, \qquad \gamma = \sqrt{\alpha^2 + \beta^2} \tag{17}$$

It follows from (2), (9), (12), (14) and (17) that for $x_1 = 0, a$ and $x_2 = 0$ the hinge-support conditions are satisfied in an integral sense, i. e.

$$\begin{aligned} u_3|_{x_1=0,\,a} = 0, \quad P_1|_{x_1=0,\,a} = 0, \quad \int_{-h}^{+h} P_2|_{x_1=0,\,a}\, dx_3 = 0 \\ u_3|_{x_2=0,\,b} = 0, \quad P_2|_{x_2=0,\,b} = 0, \quad \int_{-h}^{+h} P_1|_{x_2=0,\,b}\, dx_3 = 0 \end{aligned} \tag{18}$$

Substituting the solution (17) into the boundary conditions (16), and taking account of (2) and (12), as a result of the customary procedure we obtain a transcendental equation to determine the critical value of p, which becomes after manipulation

$$\begin{aligned} &\frac{1}{\zeta_2}\left[a_{33}\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_2^2\right)+a_{13}\zeta_2^2\right]\left[\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_3^2\right)-1\right]\times \\ &\times \operatorname{sh}\frac{\gamma h}{\zeta_2}\operatorname{ch}\frac{\gamma h}{\zeta_3}-\frac{1}{\zeta_3}\left[a_{33}\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_3^2\right)+a_{13}\zeta_3^2\right]\times \\ &\times\left[\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_2^2\right)-1\right]\operatorname{sh}\frac{\gamma h}{\zeta_3}\operatorname{ch}\frac{\gamma h}{\zeta_2}=0 \end{aligned} \tag{19}$$

Let us analyze the roots of (19) for thin-walled plates, when $\gamma h \ll 1$. Utilizing this condition, we assume approximately

$$\begin{aligned} \operatorname{sh}\frac{\gamma h}{\zeta_i} &\approx \left(\frac{\gamma h}{\zeta_i}\right)\left[1+\frac{1}{6}\left(\frac{\gamma h}{\zeta_i}\right)^2+\frac{1}{120}\left(\frac{\gamma h}{\zeta_i}\right)^4\right] \\ \operatorname{ch}\frac{\gamma h}{\zeta_i} &\approx 1+\frac{1}{2}\left(\frac{\gamma h}{\zeta_i}\right)^2+\frac{1}{24}\left(\frac{\gamma h}{\zeta_i}\right)^4 \end{aligned} \tag{20}$$

Substituting (20) into (19), we obtain an equation to an accuracy of order $(\gamma h)^4$, whose solution we represent as

$$p = \sum_{i=0}^{\infty} (\gamma h)^i p_i \tag{21}$$

As a result of the customary procedure, we obtain a value of p_i. Limiting ourselves to the determination of p_0, p_1 and p_2 we obtain the final result

$$p = p^* \left[1 - \pi^2 h^2 \left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\frac{6\,(a_{11}a_{33}-a_{13}^2)+G\,(5a_{33}-2a_{13})}{15a_{33}G}\right] \tag{22}$$

Here p^* denotes the value of the Euler force for buckling in the m, n-mode when the stress-strain relationship is in the form (2)

$$p^* = \frac{1}{3}\,\frac{a_{11}a_{33}-a_{13}^2}{a_{33}}\,\pi^2 h^2 \left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right) \tag{23}$$

Evidently (22) takes on the minimal value for $m = n = 1$.

Let us examine the stability of a circular plate of radius R and thickness $2h$ compressed by stress resultants of intensity p in the x_1x_2 plane of the plate in the axisymmetric strain case. According to (13) we take the solution of the fundamental equations

in the form
$$u_r = -\frac{\partial^2}{\partial r\,\partial x_3}(\Psi_2+\Psi_3), \quad \Psi_1 \equiv 0, \quad u_\theta = 0 \tag{24}$$
$$u_3 = \frac{a_{11}-p}{a_{13}+G}\left[\left(\frac{G}{a_{11}-p}-\zeta_2^2\right)\frac{\partial^2}{\partial x_3^2}\Psi_2+\left(\frac{G}{a_{11}-p}-\zeta_3^2\right)\frac{\partial^2}{\partial x_3^2}\Psi_3\right]$$
$$\Psi_i = \sum_{k=0}^{\infty} A_k^{(i)} J\left(\frac{\varkappa_k}{R}r\right)\operatorname{ch}\frac{\varkappa_k}{R\zeta_i}x_3 \qquad (i=2,\,3)$$

According to (14) the boundary conditions for $x_3 = \pm h$ are
$$\sigma_{33} = 0,\ \sigma_{r3} = 0,\ \text{for}\ x_3 = \pm h \tag{25}$$

According to (14) we represent the external loading components at $r = R$ as
$$P_r = (\sigma_{rr} + p u_{r,r}),\ \ P_3 = (\sigma_{r3} + p u_{3,r}) \tag{26}$$

Let us consider the hinge-support case for a rectangular plate, and the rigid fixing case for a circular plate. We hence assume that $\varkappa_k$ are the roots of Eq.
$$J'(\varkappa_k) = 0 \tag{27}$$

It follows from (2), (9), (24) and (26), that for $r = R$ the following boundary conditions are satisfied:
$$u_r|_{r=R} = 0, \qquad \int_{-h}^{+h} P_r|_{r=R}\,dx_3 = 0, \qquad u_3|_{\substack{r=R\\x_3=0}} = 0 \tag{28}$$

We can always satisfy the third condition in (28) by adding an arbitrary constant to the displacement u_3 in (24).

Substituting the solution (24) into the boundary conditions (25), we obtain after the customary procedure, a transcendental equation to determine ζ
$$\frac{1}{\zeta_2}\left[a_{33}\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_2^2\right)+a_{13}\zeta_2^2\right]\left[\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_3^2\right)-1\right]\times$$
$$\times\operatorname{sh}\frac{\varkappa_k}{\zeta_2}\frac{h}{R}\operatorname{ch}\frac{\varkappa_k}{\zeta_3}\frac{h}{R}-\frac{1}{\zeta_3}\left[a_{33}\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_3^2\right)+a_{13}\zeta_3^2\right]\times$$
$$\times\left[\frac{a_{11}-p}{a_{13}+G}\left(\frac{G}{a_{11}-p}-\zeta_2^2\right)-1\right]\operatorname{sh}\frac{\varkappa_k}{\zeta_3}\frac{h}{R}\operatorname{ch}\frac{\varkappa_k}{\zeta_2}\frac{h}{R}=0 \tag{29}$$

Let us analyze the roots of (29) for thin-walled plates, whereupon $h/R \ll 1$. Limiting ourselves to the formation of a small number of protuberances along the radius, we obtain by analogy with (19)–(22)
$$p = \frac{1}{3}\frac{a_{11}a_{33}-a_{13}^2}{a_{33}}\left(\varkappa_k\frac{h}{R}\right)^2\left[1-\left(\varkappa_k\frac{h}{R}\right)^2\frac{6(a_{11}a_{33}-a_{13}^2)+G(5a_{33}-2a_{13})}{15a_{33}G}\right] \tag{30}$$
which takes its minimal value for $k = 1$, hence
$$p_* = p^*\left[1-14.68\left(\frac{h}{R}\right)^2\frac{6(a_{11}a_{33}-a_{13}^2)+G(5a_{33}-2a_{13})}{15a_{33}G}\right], \quad \varkappa_1 \approx 3.83 \tag{31}$$
where p^* denotes the value of the Euler force
$$p^* = 14.68\left(\frac{h}{R}\right)^2\frac{1}{3}\frac{a_{11}a_{33}-a_{13}^2}{a_{33}}$$

Therefore, as a result of analyzing (22) and (31) we arrive at the deductions:

1. The Kirchhoff-Love hypothesis is asymptotically exact in the theory of plate stability.

2. The theory constructed by using the Kirchhoff-Love hypothesis yields a high value

of the critical force.

These deductions have been obtained independently of the properties of the plate material, and refer equally to both isotropic, and transversely isotropic materials with small shear stiffness, and to both nonlinearly elastic and elastoplastic materials if the concept of a tangent modulus is assumed in the latter cases.

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STATISTICAL DETERMINATION OF THE TENSOR OF VISCOSITY COEFFICIENTS

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An expression for the tensor of viscosity coefficients in the terms of the autocorrelation functions is obtained under the assumptions of the Kubo theory [1] of the linear reaction of a system subjected to a mechanical perturbation. These coefficients are obtained as components of a fourth rank tensor for an arbitrary homogeneous anisotropic medium using the framework of the Gibbs formalism without, however, employing the well-known additional representations. Coefficient of the shear viscosity of an isotropic medium is determined to illustrate the proposed method of computing the integrals of autocorrelation functions. This method uses the concept of statistical averaging in the state of equilibrium and utilizes mean relaxation times. Double index correlative distribution functions are used to obtain, by statistical methods, the relaxation times for the impulse-dependent quantities and for the spatial coordinate-dependent magnitudes. Numerical estimates for simple fluids show, that the impulse relaxation time is of the order of 10^{-14} sec, while the coordinate relaxation time is of the order of 10^{-12} sec.